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LARRY SHEPP
and
our three problems

I met Larry in 1966 at the World Mathematical Congress in Moscow. Lester Dubins and I were sitting together during the General Opening procedure in the Kremlin Palace of Congresses. Looking at a walking imposing man, Lester asked me: “Do you know this guy?”

—No,—I said.

—You should know him, he is very interesting and you have many interesting things in common, indeed.

It was beginning of our contacts and we decided that our more close connections and work would be during his planned visit to Moscow in the frame of the official contract between our Academies.

When he arrived for the second time, we began to discuss different problems. He wanted also to discuss the different styles of our lives, the differences between our counties. But my answer was from the beginning the following: “Larry, I am ready to discuss with you different mathematical problems but the rest is ours and let us not discuss it.” Larry accepted my point of view and we did not discuss the “political” problems although sometimes we had “political” discussions.

At the beginning of our contacts I told to Larry my results about Bayesian formulations of the disorder problems which greatly impressed him. He said later many times that the method of the free-boundary problems, smooth-fit conditions, and innovation Wiener processes he got from me, and we used them several times in our work.

Now I consider here one our works together with Lester Dubins who had some results with G. Schwarz in 1988 (“A sharp inequality for sub-martingale and stopping times”, Astérisque 157-158 (1988), 129–145). Larry and I were very unhappy with their arguments. Then Lester said: “Give your proof.” After some time it was done in our paper

(I) “Optimal stopping rules and maximal inequalities for Bessel process”

(Theory Probab. Appl. 38:2 (1993), 226-261).

The problem (in simple version) was the following: To find

$$\boxed{\sup_{\tau} \mathsf{E} \max_{t \le \tau} |B_t|, \quad \textbf{where } B = (B_t)_{t \ge 0} \textbf{ is a Brownian motion}}\,.$$

If $\tau \equiv T$ is a deterministic time, then $\mathsf{E}\max_{s \le T} |B_s|$ is easy to find:

$$\mathsf{E} \max_{s \le T} |B_s| = \sqrt{\frac{\pi}{2} T} = 1.25331 \ldots \sqrt{T}$$

(at the same time, by the way, $\mathsf{E}|B_T| = \sqrt{(2/\pi)T} = 0.79788\ldots\sqrt{T}$).

It is clear of course that $\mathsf{E}\max_{s \le T} |B_s|$ can be found from the famous formula for distribution $\mathsf{P}(\max_{s \le T} |B_s| \le x)$. But it is not simple because the corresponding expression for this law is given by $\underline{\underline{\text{sign-changing series}}}$ that gives some difficulties for changing the order of integration (in $\int_0^\infty x \, d\mathsf{P}(\max_{s \le T} |B_s| \le x)$.

However, here we may operate in a different way.

Indeed, take $T = 1$ and using the properties of the Brownian motion we easily find that

$$\mathsf{P}\Big(\sup_{t\le 1}|B_t| \le x\Big) = \mathsf{P}\Big(\frac{1}{\sqrt{\sigma}} \le x\Big),$$

where $\sigma = \inf\{t \ge 0\colon |B_t| = 1\}$. So,

$$\sup_{t\le 1}|B_t| \overset{\text{law}}{=} \frac{1}{\sqrt{\sigma}}.$$

For a normal distribution

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \mathsf{E}e^{-x^2/(2\sigma^2)}\,dx = a, \qquad a > 0.$$

So,

$$\mathsf{E}\sup_{t\le 1}|B_t| = \mathsf{E}\frac{1}{\sqrt{\sigma}} = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathsf{E}e^{-x^2\sigma/2}\,dx.$$

Using the well-known Laplace transform

$$\mathsf{E}e^{-\lambda\sigma} = \frac{1}{\cosh\sqrt{2\lambda}},$$

we get

$$\begin{aligned}\mathsf{E}\sup_{t\le 1}|B_t| &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{dx}{\cosh\sqrt{x}} = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^x\,dx}{e^{2x}+1} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{dy}{1+y^2} \\ &= 2\sqrt{\frac{2}{\pi}}\arctan x\Big|_1^\infty = 2\sqrt{\frac{2}{\pi}}\,\frac{\pi}{4} = \sqrt{\frac{2}{\pi}}.\end{aligned}$$

From here we find that

$$\mathsf{E}\sup_{t\le T}|B_t| = \sqrt{\frac{2}{\pi}T}.$$

But, of course, the problem of finding $\mathsf{E}\sup_{s\le\tau}|B_s|$ with τ random is much more difficult.

In our work (I) we used the following method.

Consider the optimal stopping problem

$$V(c) = \sup_{\tau} \mathsf{E}\Big[\max_{t\le\tau}|B_t| + c\tau\Big] \quad \text{with } c>0.$$

If we can find $V(c)$, then

$$\boxed{\mathsf{E}\max_{t\le\tau}|B_t| \le V(c) + c\mathsf{E}\tau} \quad \text{and} \quad \boxed{\mathsf{E}\max_{t\le\tau}|B_t| \le \inf_{c>0}[V(c) + c\mathsf{E}\tau]}.$$

To find $V(c)$ we use the method of the "Stefan problem" or "free-boundary problem" for 2-dimensional Markov process $(B_t, \max_{s\le t}|B_s|)_{t\ge 0}$. As a result we find that $\inf_{c>0}[V(c)+c\mathsf{E}\tau] = \sqrt{2\mathsf{E}\tau}$. So,

$$\boxed{\mathsf{E}\max_{t\le\tau}|B_t| \le \sqrt{2\mathsf{E}\tau}}.$$

(It is possible to show that this is a sharp inequality.)

Another problem we worked with Larry is the so-called

(II) "Russian option".

This financial problem consists in finding

$$V = \sup_{\tau} \mathsf{E} e^{-(r+\lambda)\tau} M_\tau,$$

where $M_t = \max_{u \le t} S_u$ and $S = (S_t)_{t \ge 0}$ is a geometrical Brownian motion with

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t.$$

In our paper

(II) **"The Russian option: Reduced regret"**

(Ann. Appl. Probab. 3:3 (1993), 631-640)

we demonstrated that optimal stopping time is given by

$$\boxed{\tau = \inf\{t \geq 0\colon X_t \geq b\}, \quad \text{where } X_t = \frac{\max_{u\leq t} S_u}{S_t}}.$$

It is interesting that the initial problem is **two**-dimensional optimal stopping problem for $(S_t, \max_{u\leq t} S_u)$. In this case this **two**-dimensional process is Markov process. However, optimal stopping problem ($X_t = (\max_{u\leq t} S_u)/S_t$) is **one**-dimensional.

In our next paper

“A new look at pricing of the «Russian option»”

(Theory Probab. Appl. 39:1 (1994), 103-119)

we gave another method which used a change of measure. As a result we obtained the one-dimensional optimal stopping problem which gives a better explanation of the “one-dimensional” form of optimal stopping time:

$$\boxed{\tau = \inf\left\{t \geq 0 \colon \frac{\max_{u \leq t} S_u}{S_t} \geq b\right\}}$$

The third our problem was published in the paper

(III) "Hiring and firing optimally in a large corporation"

(J. Economic Dynamics and Control 20 (1996), 1523-1539).

This paper assumes that capital of a firm is a process $X = (X_t)_{t \geq 0}$ which has a stochastic differential

$$\boxed{dX_t = -dZ_t + U_t(\mu\, dt + \sigma\, dW_t) - K_-\, d_-U_t - K_+\, d_+U_t},$$

where • U_t is "size" of a firm: $dU_t = d_+U_t - d_-U_t$ with $d_+U_t, d_-U_t \geq 0$,

• $Z = (Z_t)_{t \geq 0}$ are dividends.

We want to find $$V(x,u) = \sup_{(Z,U)} \mathsf{E}_{x,u} \int_0^\tau e^{-\lambda t}\, dZ_t,$$
where τ is a time of bankruptcy.

In the paper we described the optimal "hiring and firing procedure", which is sufficiently difficult. The discussions with Roy Radner were here very useful for formulation of a problem.

Here I described only three our works with Larry. We considered, of course, other problems, for example, the optimal stopping problem for non-Markov times. Our last problems were related with our project “Stochastic calculus in Medical Science” but it was not realized. Too pity!